Perfectoid Spaces

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$\boxed{\text{Recall}}$: (A, A^+) perfectoid affinoid

multiplicative, cont., but not additive in general

- $\varprojlim_{x \mapsto x^p} A = A^\flat$, where $(a_i) + (b_i) = \left(\lim_{j \to \infty} (a_{i+j} + b_{i+j})^{p^j} \right)$, $\#: A^\flat \to A$ proj. on first comp.

  $\hookrightarrow \operatorname{Im} \# \subseteq A$ dense

$\boxed{\text{Facts}}$:

1. $A^\flat$ perfectoid Tate
2. $(A^\flat)^\circ = (A^\circ)^\flat = \varprojlim_{F} A^\circ / \varpi$
3. $\exists\, \varpi \in A$ ps-unif, $\varpi^p | p$ and $\varpi = \#(\varpi^\flat)$ for some $\varpi^\flat \in A^\flat$
4. if ϖ is as above, then
   - (4a) $A^\flat = A^{\flat\circ}[\frac{1}{\varpi^\flat}]$
   - (4b) $A^{\flat\circ}/\varpi^\flat \cong A^\circ/\varpi$
5. $A^{\flat +} = \varprojlim_{x \mapsto x^p} A^+ \overset{(5a)}{\Longrightarrow} (A^\flat, A^{\flat +})$ perf. affinoid

   (5b) $$\begin{array}{ccc} A^{\flat\circ}/\varpi^\flat & \cong & A^\circ/\varpi \\ \cup\!\!\mid & & \cup\!\!\mid \\ A^{\flat +}/\varpi^\flat & \cong & A^+/\varpi \end{array}$$ (for ϖ as in (3))

6. $|\cdot| \in \operatorname{Cont}(A)$, $|a|^\flat := |a^\#| \Rightarrow |\cdot|^\flat \in \operatorname{Cont}(A^\flat)$
7. $\phi: W(A^{\flat +}) \xrightarrow[\text{surj.}]{\text{Ring. hom.}} A^+$, $\ker \phi = (p + [\varpi]_{\text{Teich}} \cdot \underset{W(A^{\flat+})}{\alpha})$

   $\sum [a_i]_{\text{Teich}}\, p^n \longmapsto \sum a_i^\# p^n$

$\boxed{\text{Almost mathematics}}$

Recall that $A^{\circ\circ} \subseteq A^+$. Let M be a module over A^+.

$\boxed{\text{Def}}$: M almost 0 if $A^{\circ\circ} \cdot M = 0$ ($A^{\circ\circ} \cdot A^{\circ\circ} = A^{\circ\circ} \rightsquigarrow$ Serre subcategory).

$\boxed{\text{Ex}}$: For $\widehat{\mathbb{Z}_p[p^{1/p^\infty}]}^p$, you want $p^{1/p^j} \cdot M = 0$ $\forall j$.

$\boxed{\text{Def}}$: To come back from the almost side, $M_! := A^{\circ\circ} \otimes_{A^+} M$ ('without'), $M_* = \operatorname{Hom}(A^{\circ\circ}, M)$ ('included')

$\boxed{\text{Claim}}$: $M \to M_*$, $m \longmapsto (a \mapsto a \cdot m)$ almost isom.

$\boxed{\text{Pf}}$: $0 \to A^{\circ\circ} \to A^+ \to A^+/A^{\circ\circ} \to 0 \xrightarrow{\operatorname{Hom}(-, M)}$ $\square$

$\boxed{\text{Setting}}$ (Investigate Spa for A & $A^\flat$):

$$\begin{array}{ccc}
(A, A^+) & \xleftarrow{\#} & (A^\flat, A^{\flat +}) \\
\varpi & \longleftarrow & \varpi^\flat \\
f_i & \longleftarrow & f_i^\flat \\
g & \longleftarrow & g^\flat \\
f_n = \varpi^N & \longleftarrow & f_n^\flat = (\varpi^\flat)^N
\end{array}
\quad \rightsquigarrow \quad
\begin{array}{ccc}
R\left(\frac{f_1, \dots, f_n}{g}\right) = U & \xrightarrow{\flat} & U^\flat = R\left(\frac{f_1^\flat, \dots, f_n^\flat}{g^\flat}\right) \\
\cap\!| & & \cap\!| \\
\mathrm{Spa}(A, A^+) = X & & X^\flat = \mathrm{Spa}(A^\flat, A^{\flat +})
\end{array}$$

$\boxed{\text{Thm}}$:

① $(\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ perfectoid affinoid, and $(\mathcal{O}_X(U), \mathcal{O}_X^+(U))^\flat = (\mathcal{O}_{X^\flat}(U^\flat), \mathcal{O}_{X^\flat}^+(U^\flat))$

② $\mathcal{O}_X(U)^+ = \left(\widehat{A^+\langle (\frac{f_i}{g})^{1/p^\infty} \rangle}^{\varpi}\right)_*$

$\boxed{\text{Thm}}$: (A, A^+) perfectoid affinoid

① $X = \mathrm{Spa}(A, A^+) \xrightarrow[\text{homeo}]{\flat} \mathrm{Spa}(A^\flat, A^{\flat +})$, identifying rational open sets

② as ① for the previous thm.

$\boxed{\text{Thm}}$: (A, A^+) perfectoid affinoid

① $\varphi: A \to B$ finite étale, B^+ int closure of A^+ in B $\Rightarrow$ (B, B^+) perfectoid affinoid

② $\left\{ \begin{array}{c} \text{finite étale covers} \\ \text{of } (A, A^+) \end{array} \right\} \xleftrightarrow{\text{tilting}} \left\{ \begin{array}{c} \text{finite étale covers} \\ \text{of } (A^\flat, A^{\flat +}) \end{array} \right\}$

$\boxed{\text{Pf of 'B is perfectoid in ①'}}$ (outline):

$\boxed{a}$ Show the statement for A a field

$\boxed{b}$ Look at $\flat: X \to X^\flat$ and fix $x \in X$.

$\boxed{\text{Lemma}}$: $\mathcal{O}_{X,x} \xrightarrow{\cong} k(x)$ after ϖ-adic completion.

note: $|\cdot|_x \in \mathrm{Spa}(A, A^+)$
$\forall f_i, g$ s.t. $x \in R(\frac{f_i}{g})$: $|\cdot|_x \in \mathrm{Spa}(A\langle \frac{f_i}{g} \rangle, A\langle \frac{f_i}{g} \rangle^+)$
$\rightsquigarrow |\cdot|_x \in \mathrm{Spa}(\mathcal{O}_{X,x}, \mathcal{O}_{X,x}^+)$ (direct limit)
$\hookrightarrow \mathfrak{m}_{X,x} = \{a \in \mathcal{O}_{X,x} \mid |a|_x = 0\}$. Indeed, if $b \in \mathcal{O}_{X,x}$ s.t. $|b|_x \neq 0$. Note $\exists f_i, g$ such that $b \in A\langle \frac{f_i}{g} \rangle$, $x \in R(\frac{f_i}{g})$ $\rightsquigarrow$ find smaller $R(\dots)$ on which b invertible
$\hookrightarrow b \in \mathcal{O}_{X,x}^*$. Also: $\mathcal{O}_{X,x}^+ = \mathcal{O}_{X,x}^{|\cdot|_x \leq 1}$

$\boxed{c}$
$$\begin{array}{ccccc}
(B, B^+) & \longrightarrow & (L, L^+) & \overset{\varpi}{\rightsquigarrow} & (L^\flat, \dots) \\
\uparrow & & \uparrow & & \uparrow \text{étale} \\
(A, A^+) & \longrightarrow & (k(x), k(x)^+) & & (k(x), k(x)^+)
\end{array} \quad \dots \square$$

$\boxed{\text{How to apply ?}}$:

$$\begin{array}{ccc}
Z & \longleftarrow & Z_K \\
\downarrow {\scriptstyle \text{sm. proj. var.}} & & \downarrow \\
\mathrm{Spec}\,\mathbb{Q}_p & \longleftarrow & \mathrm{Spec}\,K \\
 & & \| \\
 & & \widehat{\mathbb{Q}_p[p^{1/p^\infty}]}^p
\end{array}$$

How to turn Z_K into something perfectoid? If $Z = \mathbb{P}^n$, one has
$\mathbb{P}^n_K \longrightarrow \varprojlim_{x_i \mapsto x_i^p} \mathbb{P}^n_K = W_K$ locally $\mathrm{Spa}\, K\langle (\frac{x_0}{x_i})^{1/p^\infty}, \dots, (\frac{x_n}{x_i})^{1/p^\infty} \rangle_p$

$\hookrightarrow \begin{array}{c} W_{K^\flat}^\flat \\ \downarrow \\ \mathrm{Spec}\,K^\flat \end{array}$